International Journal of Theoretical Physics, Vol. 36, No. 6, 1997

Multiple Singular Manifold Method and Extended Direct Method: Application to the Burgers Equation

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Received June 7, 1996

This paper considers the relationship between the multiple singular manifold method (MSMM) and the extended direct method (EDM) for studying partial differential equations. It is shown that the similarity reductions using EDM can be obtained by MSMM. The prototype example for illustrating the approach is the Burgers equation, which is the simplest evolution equation to embody nonlinearity and dissipation. As a conclusion of the MSMM, we obtain a set of Bäcklund transformations of the Burgers equation.

1. INTRODUCTION

In the study of partial differential equations (PDEs), the discovery of exact or special solutions has great theoretical and practical importance. Recently several new methods for obtaining similarity reductions for PDEs have been developed, as follows:

(a) The symmetry group method. The classical Lie group method for finding symmetry reductions of PDEs was first introduced by Lie (1881). Bluman and Cole (1969) extended Lie's reduction method to include nonclassical symmetry groups, where the invariance of PDEs is only required on its intersection with the invariance surface condition characterizing the group functions, which is the so-called nonclassical symmetry method or the method of conditional symmetry. Later, Olver and Rosenau (1986) proposed an extension of the nonclassical method, but their framework appears to be too general to be practical. Recently, Fokas and Liu (1994) introduced the concept of generalized conditional symmetry, which can be applied to derive new reductions for a class of PDEs (Qu, 1996).

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(b) The direct method. Clarkson and Kruskal (1989) introduced an algorithmic method which reduces a PDE to a single ordinary differential equation (ODE) (Hereafter, we refer to this as the direct method to distinguish it from the EDM), and which was developed and subsequently applied to many PDEs arising in a wide variety of physical systems. More recently, Hood (1995) extended the direct method to a more general case; we refer to this as the extended direct method (EDM). The idea is to seek a transformation which reduces the given PDE to a system of ODEs in $p(\xi)$ and $q(\eta)$ by means of an ansatz

$$u(x, t) = \alpha(x, t) + \beta(x, t)p(\xi) + \gamma(x, t)q(\eta) \tag{1}$$

other than the original ansatz due to Clarkson and Kruskal (1989)

$$u(x, t) = \alpha(x, t) + \beta(x, t)p(\xi) \tag{2}$$

for the Burgers equation

$$u_t + uu_x + u_{xx} = 0 \tag{3}$$

The EDM will enable a wide class of new solutions to be found.

(c) The singular manifold method. The generalization of the *Painlevé* analysis to partial differential equations without referring to ODEs has been formulated by Weiss *et al.* (1983) and then developed further in series of papers (Weiss, 1983, 1984, 1985). Roughly speaking, a PDE possesses the *Painlevé* property if its solutions are single-valued about movable singularity manifolds. The approach requires that solutions of the PDE can be written in *Painlevé* series

$$u(x, t) = \sum_{0}^{\infty} u_j(x, t)\phi^{j-\alpha} \tag{4}$$

where $\phi(x, t)$ is an arbitrary analytic function depending on initial conditions that will be called the movable singularity manifold. For us it is more important here to concentrate on the so-called singular manifold method, which emphasizes only the solutions which arise in the truncation of the series (4) as

$$u(x, t) = \sum_{0}^{n} u_j(x, t)\phi^{j-\alpha} \tag{5}$$

Recently Estevez *et al.* (1993; Gordoa and Estevez, 1994) presented a unified treatment of a modified singular manifold expansion method as an improved variant of the *Painlevé* analysis for PDEs with two branches in the *Painlevé* expansion. The solution can be expressed in the form

$$u = u_n + \sum_{0}^{n-1} u_j\phi^{j-n} + \sum_{0}^{n-1} v_j\sigma^{j-n} \tag{6}$$

We call this method the double singular manifold method. If the number of singular manifolds is n, $n \geq 2$, we refer to it as the multiple singular manifold method (MSMM).

The relationship among the above three approaches was treated in (Nucci and Clarkson, 1992; Arrigo *et al.*, 1993; Estevez *et al.*, 1992; Estevez and Gordoa, 1995). The aim of the present paper is to discuss the connection between MSMM and the EDM by using the Burgers equation as an illustrative example. The outline of this paper is as follows. In Section 2, we give a brief discussion of double singular manifold method for the Burgers equation. The relationship between the double singular manifold and the EDM is presented in Section 3. In Section 4, we obtain a set of new Bäcklund transformations of the Burgers equation by MSMM. Section 5 is a summary and discussion of our results.

2. MULTIPLE SINGULAR MANIFOLD METHOD OF BURGERS EQUATION

The solution expansion of the Burgers equation with n-singular manifold takes the form

$$u = \beta + \sum_{j=1}^{n} \frac{u_j}{\phi_j} \tag{7}$$

where $u_j, j = 1, 2, \ldots, n$, are arbitrary analytic functions in a neighborhood of $\phi_j = 0, j = 1, 2, \ldots, n$. Without loss of generality, we consider the case $n = 2$, i.e., the case of double singular manifolds. Hence (7) reduces to

$$u = \beta + \frac{u_1}{\phi} + \frac{u_2}{\sigma} \tag{8}$$

Four cases arise in terms of ϕ and σ.

2.1. $\phi_x\sigma_x \neq 0$. Inserting expansion (8) in (3), one obtains

$$u_1 = 2\phi_x, \qquad u_2 = 2\sigma_x \tag{9}$$

and ϕ, σ, β are constrained by

$$\beta = -\frac{\sigma_t}{\sigma_x} - \frac{\sigma_{xx}}{\sigma_x} = -\frac{\phi_t}{\phi_x} - \frac{\phi_{xx}}{\phi_x} - \frac{\sigma_x}{\sigma} \tag{10}$$

$$\beta_t + \beta\beta_x + \beta_{xx} = 0 \tag{11}$$

The substitution of (10) into (11) implies σ satisfying

$$\frac{\partial}{\partial t}\left(\frac{\sigma_t}{\sigma_x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\sigma_t}{\sigma_x}\right)^2 + 2\frac{\partial^2}{\partial x^2}\left(\frac{\sigma_t}{\sigma_x}\right) + \frac{\partial}{\partial x}\{\sigma; x\} = 0 \tag{12}$$

where

$$\{\sigma; x\} = \frac{\partial}{\partial x}\left(\frac{\sigma_{xx}}{\sigma_x}\right) - \frac{1}{2}\left(\frac{\sigma_{xx}}{\sigma_x}\right)^2 \tag{13}$$

is the Schwarzian derivative.

2.2. $\phi_x = \sigma_x = 0$, i.e., ϕ and σ are characteristic manifolds. Substituting (8) into (3), we have

$$u_1 = \phi_t x + \alpha_1(t), \qquad u_2 = \sigma_t x + \alpha_2(t) \tag{14}$$

and the compatibility condition

$$\beta_t + \beta_{xx} + \beta\beta_x = 0 \tag{15}$$

$$\beta(\sigma_t x + \alpha_2) = -\frac{\sigma_{tt}}{2} x^2 - \alpha_{2t} x + \alpha_3(t) \tag{16}$$

$$\beta(\phi_t x + \alpha_1) = -\left(\frac{\phi_{tt}}{2} + \frac{\sigma_t \phi_t}{\sigma}\right)x^2 - \left(\alpha_{1t} + \frac{\alpha_1 \sigma_t}{\sigma} + \frac{\phi_t \alpha_2}{\sigma}\right)x + \alpha_4(t) \tag{17}$$

where α_i, $i = 1, 2, 3, 4$, are functions of t to be determined.

It follows from (15)–(17) that

$$\phi = D_1 + \frac{D_2}{\sigma} \tag{18}$$

with two arbitrary constants D_1 and D_2. The α_2, α_3, and α_4 can be determined by α_1.

2.3. $\phi_x \neq 0$, $\sigma_x = 0$. Inserting (8) in (3), we obtain

$$u_1 = 2\phi_x, \qquad u_2 = \sigma_t x + \alpha_2(t) \tag{19}$$

and a system of equations for β, ϕ, and σ:

$$\beta_t + \beta_{xx} + \beta\beta_x = 0 \tag{20}$$

$$\beta = -\frac{\phi_t}{\phi_x} - \frac{\phi_{xx}}{\phi_x} - \frac{\sigma_t x + \alpha_2(t)}{\sigma} \tag{21}$$

$$\beta(\sigma_t x + \alpha_2) = -\frac{\sigma_{tt} x^2}{2} - \alpha_{2t} x + \alpha_3(t) \tag{22}$$

where $\alpha_2(t)$ and $\alpha_3(t)$ are also functions of t to be determined.

2.4. $\phi_x = 0$, $\sigma_x \neq 0$. Substituting (8) into (3) and taking into account $\phi_x = 0$, $\sigma_x \neq 0$, one obtains

$$u_1 = \phi_t x + \alpha_1(t), \qquad u_2 = 2\sigma_x \tag{23}$$

with ϕ, σ, and β obeying

$$\beta_t + \beta_{xx} + \beta\beta_x = 0 \tag{24}$$

$$\beta = -\frac{\sigma_t}{\sigma_x} - \frac{\sigma_{xx}}{\sigma_x} \tag{25}$$

$$\beta(\phi_t x + \alpha_1(t)) + \frac{2\sigma_x}{\sigma} = -\frac{\phi_{tt}}{2} x^2 - \alpha_{1t} x + \alpha_3(t) \tag{26}$$

where α_1 and α_2 are two functions of t to be determined.

It is worth noting that the above analysis will play a fundamental role in discussing the relationship between MSMM and EDM.

3. DOUBLE SINGULAR MANIFOLD METHOD AND EDM

As mentioned in the Introduction, Hood (1995) proposed an extended version of the direct method for finding similarity reductions of PDEs and applied this to compute new classes of solutions of the Burgers equation by reduction to a pair of ODEs in terms of the ansatz (1). To establish the relationship of the double singular manifold method and the EDM, we impose that the double singular manifold ϕ and σ must be just functions of the reduced variables ξ and η, respectively. Then we will show that the similarity reductions of the Burgers equation using EDM can be obtained by the double singular manifold method of the previous section. Four cases corresponding to Section 2 are considered separately.

3.1. $\phi_x \neq 0$, $\sigma_x \neq 0$. We assume that the reduced variables ξ and η take the form

$$\xi = \alpha(t)x + \theta(t), \qquad \eta = \mu(t)x + \nu(t) \tag{27}$$

The substitution of (27) into (10) gives

$$\beta = -\frac{a_t x + \theta_t}{\alpha} - \alpha_t \frac{\phi_{\xi\xi}}{\phi_\xi} - 2\mu(t)\frac{\sigma_\eta}{\sigma} \tag{28}$$

To continue, we distinguish three subcases:

3.1.1. $\eta = k\xi$, $k =$ const. If we choose

$$\phi = c_1 \int_0^\xi \exp(-2\lambda_3\xi^2)\, d\xi + c_2$$
$$\sigma = c_3 \exp\left(-\int_0^\eta \frac{q(\eta)}{2}\, d\eta\right), \qquad \lambda_3 = \text{const} \tag{29}$$

Hereafter c_i, $i = 1, 2, \ldots$, denote arbitrary constants. Then (28) becomes

$$\beta = \frac{\alpha_t x + \theta_t}{\alpha} + 4\lambda_3\alpha(t)\xi + k\alpha(t)q(\eta) \tag{30}$$

which is nothing but the reduction Ansatz of III.A.1 from Hood (1995).

3.1.2. $\eta \neq k\xi$, $k = \text{const}$. Taking

$$\phi = c_4 \int_0^\xi \exp(-\lambda_3\xi^2)\, d\xi + c_5, \qquad \sigma = c_6 \exp\left(-\int_0^\eta \frac{q(\eta)}{2}\, d\eta\right) \tag{31}$$

so that

$$\beta = -\frac{\alpha_t x + \theta_t}{\alpha} + 2\lambda_3\alpha(t)\xi + \mu(t)q(\eta) \tag{32}$$

which, combined with (10), gives the reduction III.A.2 of Hood (1995).

3.1.3. The reduction III.A.3 of Hood (1995) can be obtained by taking

$$\phi = -\frac{c_7}{\xi} + c_8, \qquad \sigma = c_9 \exp\left(-\int_0^\eta \frac{q(\eta)}{2}\, d\eta\right) \tag{33}$$

3.2. $\phi_x = \sigma_x = 0$. Without loss of generality, one supposes the reduced variables in this case be

$$\xi = \eta = t \tag{34}$$

Setting

$$\phi = c_{10} \int_0^t \exp\left(-2 \int_0^t p(\tau)\, d\tau\right) dt + c_{11}, \qquad \sigma = c_{12} \exp\left(\int_0^t q(\tau)\, d\tau\right) \tag{35}$$

From (17), it follows that

$$\beta = \beta_1(x, t) + p(t)x + (x + \alpha_2(t))q(t) \tag{36}$$

where

$$\beta_1(x, t) = \frac{\alpha_1\phi_{tt}}{2\phi_t^2} - \frac{\alpha_1^2\phi_{tt}}{2\phi_t^2(\phi_t x + \alpha_1)} - \frac{\alpha_{1t}x + \alpha_3}{\phi_t x + \alpha_1} \tag{37}$$

Equation (36) is just the same as the reduction ansatz of III.B.1 from Hood (1995), where $p(t)$ and $q(t)$ satisfy

$$p_t + p^2 = 0, \qquad q_t + q^2 + \frac{2}{t}q = 0 \tag{38}$$

3.3. $\phi_x \neq 0$, $\sigma_x = 0$. To obtain the reductions of III.D.1 and III.D.2 of Hood (1995), we consider two subcases:

3.3.1. $\phi(\xi) = 2/\xi$. The reduced variables are $\xi = x + \lambda_1 t$, $\eta = t$, and $\lambda_1 = \text{const}$. Setting $\sigma = 1/t$, $\alpha_2 = -\lambda_1/t$, and $\alpha_3 = -2$ in (19), we find

$$\beta = \frac{2}{x + \lambda_1 t} + \frac{x}{t} \tag{39}$$

which is exactly the reduction III.D.1 of Hood (1995).

3.3.2. $\phi(\xi) = c_{12} \int_0^\xi \exp(-(\lambda_1/2)\xi^2)\, d\xi + c_{13}$, $\lambda_1 = \text{const}$. The reduced variables are $\xi = \alpha(t)x + \theta(t)$, $\eta = t$. We impose

$$\sigma = \exp\left(-\int_0^t q(\tau)\, d\tau\right) \tag{40}$$

Substitution of ϕ and (40) into (21) yields

$$\beta = \left(\lambda_1\alpha^2 - \frac{\alpha_t}{\alpha}\right)x + \lambda_1\alpha\theta - \frac{\theta_t}{\alpha} + q(t)(x + \alpha_4(t)) \tag{41}$$

where

$$\alpha_4(t) = \frac{\alpha_2(t)}{\sigma_t} \tag{42}$$

Equation (41) gives the reduction Ansatz III.D.2. of Hood (1995). Then $q(t)$ can be determined from (22) directly.

3.4. $\phi_x = 0$, $\sigma_x \neq 0$. Taking reduced variables

$$\eta = \mu(t)x + \nu(t) \tag{43}$$

and

$$\sigma = \exp\left(-\frac{1}{2}\int_0^\eta q(\eta)\, d\eta\right), \qquad \alpha_1 = \frac{1}{t}, \qquad \alpha_3 = 0 \tag{44}$$

in (23). Then from (23) it follows that

$$\beta = \frac{x}{t} + \mu(t)q(\eta) \tag{45}$$

which is the same as the reduction III.C.1 of Hood (1995).

4. NEW BÄCKLUND TRANSFORMATIONS FOR THE BURGERS EQUATION

Let us now construct some new Bäcklund transformations of the Burgers equation (3) in terms of the MSMM. We first recall the known auto-Bäcklund transformations of the Burgers equation. Perform a truncated Painlevé expansion for u in the form

$$u = \beta + \frac{\phi_x}{\phi} \tag{46}$$

where u and β satisfy the Burgers equation (3) and

$$\phi_t + \phi_{xx} + \beta\phi_x = 0 \tag{47}$$

If $\beta = 0$, (46) and (47) give the well-known Cole–Hopf (Cole, 1951; Hopf, 1950) transformation. When $\beta = \phi$, we find the following auto-Bäcklund transformation:

$$u = \phi + \frac{2\phi_x}{\phi} \tag{48}$$

where ϕ also satisfies the Burgers equation (3).

The Bäcklund transformation (48) was first discovered by Fokas (1979) using the method of Lie symmetries. The general form of the auto-Bäcklund transformation (46) and (47) was pointed out in Weiss *et al.* (1983). to obtain new Bäcklund transformations of the Burgers equation, we use the truncated multiple singular manifold expansion for u, namely

$$u = \beta + \sum_{i=1}^{n} \frac{2\phi_{ix}}{\phi_i} \tag{49}$$

where $\beta = \beta(x, t)$ and $\phi_i(x, t)$, $i = 1, 2, \ldots, n$, are analytic functions of (x, t) in a neighborhood of the manifold $\{\phi_i = 0, i = 1, 2, \ldots, n\}$. Inserting expansion (49) in equation (3), we obtain the following constraints on ϕ_i and β:

$$\beta_t + \beta_{xx} + \beta\beta_x = 0 \tag{50}$$

$$\begin{aligned}
\phi_{nt} + \phi_{nxx} + \beta\phi_{nx} &= 0\\
\phi_{n-1,t} + \phi_{n-1,xx} + \left(\beta + \frac{2\phi_{nx}}{\phi_n}\right)\phi_{n-1,x} &= 0\\
\phi_{n-2,t} + \phi_{n-2,xx} + \left(\beta + \frac{\phi_{n-1,x}}{\phi_{n-1}} + \frac{2\phi_{nx}}{\phi_n}\right)\phi_{n-2,x} &= 0\\
&\cdots
\end{aligned}$$

$$\phi_{1t} + \phi_{1xx} + \left(\beta + 2\sum_{j=2}^{n} \frac{\phi_{jx}}{\phi_j}\right)\phi_{1x} = 0 \tag{51}$$

Solving the first of equation (51) for β and substituting it into (50), we find that ϕ_n satisfies equation (12).

If we assume $\beta = 0$ in (4.5) and (4.6), we obtain the generalized Cole–Hopf transformation

$$\phi_{nt} + \phi_{nxx} = 0 \tag{52}$$

$$\phi_{n-1,t} + \phi_{n-1,xx} + \frac{2\phi_{nx}}{\phi_n}\phi_{n-1,x} = 0$$

$$\phi_{n-2,t} + \phi_{n-2,xx} + \left(\frac{2\phi_{n-1,x}}{\phi_{n-1}} + \frac{2\phi_{nx}}{\phi_n}\right)\phi_{n-2,x} = 0$$

$$\cdots$$

$$\phi_{1t} + \phi_{1xx} + \left(2\sum_{j=2}^{n}\frac{\phi_{jx}}{\phi_j}\right)\phi_{1x} = 0 \tag{53}$$

Comparing the first of equations (53) with (52), we find ϕ_{n-1} also satisfies equation (12). When $n = 2$, another form of Bäcklund transformation (48) is found for the Burgers equation

$$u = \frac{2\phi_{2x}}{\phi_2} + \frac{2\phi_{1x}}{\phi_1} \tag{54}$$

Hence ϕ_2 and ϕ_1 solve (52) and (12), respectively.

If we impose $\beta = \phi_n$ in (50) and (51), a generalized auto-Bäcklund transformation for (48) is obtained:

$$\phi_{nt} + \phi_{nxx} + \phi_n\phi_{nx} = 0 \tag{55}$$

$$\phi_{n-1,t} + \phi_{n-1,xx} + \left(\phi_n + \frac{2\phi_{nx}}{\phi_n}\right)\phi_{n-1,x} = 0$$

$$\phi_{n-2,t} + \phi_{n-2,xx} + \left(\phi_n + \frac{2\phi_{n-1,x}}{\phi_{n-1}} + \frac{2\phi_{nx}}{\phi_n}\right)\phi_{n-2,x} = 0$$

$$\cdots$$

$$\phi_{1t} + \phi_{1xx} + \left(\phi_n + 2\sum_{j=2}^{n}\frac{\phi_{jx}}{\phi_j}\right)\phi_{1x} = 0 \tag{56}$$

Solving the first of equations (56) for ϕ_n and substituting it into (55), a detailed calculation shows that ϕ_{n-1} satisfies a trilinear PDE

$$h_x^2h_{tt} - 2h_{xt}h_th_x - 2h_x^2h_{xxt} - 4h_xh_{xx}h_{xt} - 2h_xh_th_{xxx} + 3h_th_{xx}^2 + h_x^2h_{xxxx} - 4h_xh_{xx}h_{xxx} + 3h_{xx}^3 + 2h_t^2h_{xx} = 0 \tag{57}$$

Equation (57) is a new integrable trilinear PDE to us. If $n = 2$, we find a Bäcklund transformation which relate the Burgers equation and (57), namely

$$u = \phi_2 + \frac{2\phi_{2x}}{\phi_2} + \frac{2\phi_{1x}}{\phi_1} \tag{58}$$

where u and ϕ_2 satisfy the Burgers equation, and ϕ_1 solves (57).

5. SUMMARY AND DISCUSSION

In this paper we have reviewed and compared the MSMM developed by Estevez (1992; Estevez and Gordoa, 1995) and the EDM due to Hood (1995) as techniques for determining similarity reductions of nonlinear PDEs, using the Burgers equation as an illustrative example. A detailed analysis shows that the similarity reductions obtained by the EDM can also be derived by the MSMM, i.e., the MSMM is more general than the EDM. Some new Bäcklund transformations which relate the Burgers equation and other interesting integrable nonlinear PDEs are obtained using the MSMM.

Finally, we would like to point out some open problems.

(i) In this paper, we have established the relationship between the MSMM and the EDM; we do not know whether all this analysis will eventually become a set of theorems.

(ii) Estevez and Gordoa (1995) discussed the relationships among the single singular manifold method, the direct method due to Clarkson and Kruskal (1989), and the nonclassical method due Bluman and Cole (1969). It is particularly interesting to establish a connection among the EDM, the MSMM, and the conditional symmetry method.

(iii) It would be of interest to determine whether the MSMM can be used to obtain new Bäcklund transformations of other, integrable and nonintegrable nonlinear PDEs.

We hope these problems will be solved in subsequent investigations.

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